P1

PROBABILITY AND STATISTICS FOR ENGINEERS MIDTERM 1		
Code : *CVE 303* Acad.Year : *2018-19* Semester : *Spring* Date : *31.03.2019* Time : *10:40* Duration : *110 min*	Last Name : Name : Solutions Student ID : Signature :	# :
	8 QUESTIONS ON 5 PAGES TOTAL 100 POINTS	

P1. (20)	P2. (20)	P3. (20)	P4. (15)	P5. (25)		Total. (100)

1. *(15pts)* Use the following data to fill in the table of probabilities

- $P(A) = \boxed{\frac{3}{10}}$
- $P(B \cap Y) = \boxed{\frac{4}{10}}$
- $P(X \mid A) = \boxed{\frac{1}{3}}$. $\Rightarrow P(X \cap A) = \frac{1}{3} \cdot \frac{3}{10}$

(Assume $\{A, B\}$ and $\{X, Y\}$ are two partitions of the sample space.)

Prob.	A	B	Total
X	1/10	3/10	4/10
Y	2/10	4/10	6/10
Total	3/10	7/10	1

Compute the probabilities below.

- $P(X) = 4/10$
- $P(A \cap X) = 1/10$
- $P(A \cup X) = 1/10 + 2/10 + 3/10 = 6/10$
- $P(A \mid X) = \frac{1/10}{4/10} = 1/4$
- Are A and X independent? Explain.

No. If A & X were independent then $P(A \cap X) = P(A) \cdot P(X)$
but $1/10 \neq 3/10 \cdot 4/10$.

2. *(5pts)* A test has a false positive rate of P(Test Pos. | Neg.) = 10% and a false negative rate of P(Test Neg. | Pos.) = 5%. If the probability of an event occuring is P(Pos.) = 40%, then what is the probability the event occuring if the test result is positive?

(Do not simplify your answer. Note: It may help if you write a probability table...)

Prob.	Test +	Test −	Total
+	38%	2%	40%
−	6%	54%	60%
Total	44%	56%	1

$P(- \cap \text{Test} +) = P(\text{Test} + \mid -) \cdot P(-) = 6\%$

$P(+ \cap \text{Test} -) = P(\text{Test} - \mid +) \cdot P(+) = 2\%$

$$\boxed{P(+ \mid \text{Test} +) = \frac{38}{44}}$$

P2

3. *(5×2=10pts)* The discrete random variable X has the probability mass function given to the right.

x	-1	0	1	2
$p(x)$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{1}{10}$

Give the following values. (Do not simplify fractions.)

- $E[X] = (-1)\cdot\frac{2}{10} + 0\cdot\frac{3}{10} + 1\cdot\frac{4}{10} + 2\cdot\frac{1}{10} = \boxed{4/10}$

- $E[X^2] = (-1)^2\cdot\frac{2}{10} + 0^2\cdot\frac{3}{10} + 1^2\cdot\frac{4}{10} + 2^2\cdot\frac{1}{10} = 10/10 = \boxed{1}$

- $\mathrm{Var}[X] = E[X^2] - (E[X])^2 = 1 - 16/100 = \boxed{84/100}$

- $E[2X+3] = 2E[X] + 3 = 2\cdot 4/10 + 3 = \boxed{38/10}$

- $\mathrm{Var}[3X+4] = 3^2\,\mathrm{Var}[X] = \underbrace{9\cdot 84/100}$ $= \boxed{756/100}$

This is also fine as an answer.

4. *(3+3+2+1+1=10pts)* Suppose that 50 students take an exam, and the probability of each student finding a mistake on the exam is $\frac{1}{10}$.

$X \sim \text{Binomial}(50, 1/10)$

Express the following probabilities in terms of the function `pbinom(x,n,p)`$=P(X \leqslant x)$.

- The probability that **less than** 5 students find a mistake.

$P(X<5) = P(X\le 4)$ $\boxed{\text{pbinom}(4, 50, 1/10)}$

- The probability that **more than** 5 students find a mistake.

$P(X>5) = 1 - P(X\le 5)$ $\boxed{1-\text{pbinom}(5, 50, 1/10)}$

- The probability that between 2 and 10 students **(inclusive)** find a mistake.

$P(2\le X\le 10) = P(X\le 10) - P(X\le 1)$

$\boxed{\text{pbinom}(10, 50, 1/10) - \text{pbinom}(1, 50, 1/10)}$

Express the following **numbers of students** in terms of `qbinom`$(\alpha$,n,p$)= x_\alpha$.

- The number so that the probability of **less than** that many students finding a mistake is 0.05.

$P(X<x) = .05$

$P(X\le x-1) = .05$

$\boxed{x = \text{qbinom}(.05, 50, 1/10) + 1}$

- The number so that the probability of **more than** that many finding a mistake is 0.05.

$P(X>x) = .05$

$P(X\le x) = 1 - P(X>x)$

$= 1 - .05$

$= .95$

$\boxed{x = \text{qbinom}(.95, 50, 1/10)}$

5. *(10pts)* In the parts below, indicate whether X is distributed as `Binomial`, `HyperGeometric`, `Neg.Binomial`, or `Poisson`.
(Circle one. You do **not** need to identify the parameters).
You are not allowed to ask any questions about the situations described below.

- In a statistics class, on average students answer questions correctly $\frac{4}{5}$ of the time. Let X be the number of students who give the correct answer when asked before 3 get it incorrect.

  $X \sim$ `Binomial` `HyperGeometric` `Neg.Binomial` `Poisson` $\left(\begin{array}{l} r=3 \\ p=1/5 \end{array}\right)$

- In a statistics class, $\frac{4}{5}$ of the 50 students correctly answer a question. Let X be the number of students out of the first ten to answer who are correct.

  $X \sim$ `Binomial` `HyperGeometric` `Neg.Binomial` `Poisson` $\left(\begin{array}{l} M=40 \\ N=10 \\ k=10 \end{array}\right)$

- In a statistics class, on average $\frac{4}{5}$ of all students will correctly answer a question. Let X be the number of students who correctly answer the question in a class of 50 students.

  $X \sim$ `Binomial` `HyperGeometric` `Neg.Binomial` `Poisson` $\left(\begin{array}{l} n=50 \\ p=4/5 \end{array}\right)$

- In a statistics class, on average 5 students per row will correctly answer a question. Let X be the number of students in 6 rows correctly answering the question.

  $X \sim$ `Binomial` `HyperGeometric` `Neg.Binomial` `Poisson` $(\mu=30)$

6. *(5×2=10pts)* Convert between $X \sim \text{Normal}(\mu = 1, \sigma = 2)$ and standard normal (Z).
(You do not need to simplify or reduce your answers.)

- $P(1 \leqslant X \leqslant 2) = P\left(\boxed{\frac{1-1}{2} = 0} \leqslant Z \leqslant \boxed{\frac{2-1}{2} = \frac{1}{2}}\right)$

- $P(X \leqslant x) = P\left(Z \leqslant \boxed{\frac{x-1}{2}}\right)$

- $P(1 \leqslant Z \leqslant 2) = P\left(\boxed{1\cdot 2+1 = 3} \leqslant X \leqslant \boxed{2\cdot 2+1 = 5}\right)$

- $P(Z \leqslant z) = P\left(X \leqslant \boxed{z\cdot 2+1}\right)$

- $P(|Z| \leqslant 2) = P\left(\boxed{\left|\frac{X-1}{2}\right| \leq 2}\right)$

  $= P(|X-1| \leq 4) = P(-3 \leq X \leq 5)$

7. *(5×3=15pts)* Suppose that X is a continuous random variable, $X \sim \text{Normal}(\mu = 1, \sigma = 2)$.

Express the following probabilities in terms of the function `pnorm(x,μ,σ)` $=$ $P(X \leqslant x)$.

- $P(0 < X < 2)$ $= P(X < 2) - P(X < 0)$

$$\boxed{\text{pnorm}(2, 1, 2) - \text{pnorm}(0, 1, 2)}$$

- $P(|X| > 2)$ $= P(X < -2 \text{ or } X > 2)$
$= P(X < -2) + [1 - P(X < 2)]$

$$\boxed{\text{pnorm}(-2, 1, 2) + [1 - \text{pnorm}(2, 1, 2)]}$$

- $P(|X - 1| < 1)$ $= P(-1 < X - 1 < 1) = P(0 < X < 2)$

solution #1
$(X - 1) \sim \text{Normal}(0, 2)$

$$\boxed{\text{pnorm}(\underline{1, 0}, 2) - \text{pnorm}(\underline{-1, 0}, 2)}$$

solution #2
$X \sim \text{Normal}(1, 2)$

$$\boxed{\text{pnorm}(\underline{2}, 1, 2) - \text{pnorm}(\underline{0}, 1, 2)}$$

Express the following critical values in terms of the function `qnorm(α,μ,σ)` $= x_\alpha$.

- Critical x so that $P(X > x) = 0.05$.

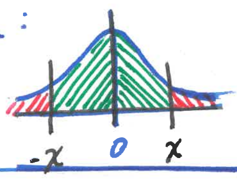
$$P(X \le x) = 1 - P(X > x)$$
$$= 1 - .05$$
$$= .95$$

$$\boxed{x = \text{qnorm}(.95, 1, 2)}$$

- Critical x so that $P(|X - 1| > x) = 0.05$.

Recall: $(X - 1) \sim \text{Normal}(0, 2)$

pdf for $X - 1$:



These two are equal (by symmetry)

$$P(|X - 1| > x) = P(X - 1 < -x) + P(X - 1 > x)$$

so $P(X - 1 < -x) = .05/2 = .025$

$P(X - 1 > x) = .05/2 = .025$

Solution #1:

$P(X - 1 < -x) = .025$

$$\boxed{x = -\text{qnorm}(.025, 0, 2)}$$

Solution #2:

$P(X - 1 < x) = 1 - P(X - 1 > x) = .975$

$$\boxed{x = \text{qnorm}(.975, 0, 2)}$$

8. *(25pts)* The discrete joint random variable (X,Y) has joint pmf given to the right.

Compute the following.

$p(x,y)$	$x=-1$	$x=0$	$x=1$	$P_Y(y)$
$y=-1$	$\frac{3}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$8/20$
$y=0$	$\frac{3}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$5/20$
$y=1$	$\frac{1}{20}$	$\frac{4}{20}$	$\frac{2}{20}$	$7/20$
$P_X(x)$	$7/20$	$8/20$	$5/20$	1

- The marginal pmf for X and Y. (Do not simplify fractions.)

x	-1	0	1
$p_X(x)$	$7/20$	$8/20$	$5/20$

y	-1	0	1
$p_Y(y)$	$8/20$	$5/20$	$7/20$

- The conditional pmf for $(X\,|\,Y=0)$ and $(Y\,|\,X=0)$. (Do not simplify fractions.)

x	-1	0	1
$p_{X\|Y}(x\,\|\,0)$	$3/5$	$1/5$	$1/5$

y	-1	0	1
$p_{Y\|X}(y\,\|\,0)$	$3/8$	$1/8$	$4/8$

Compute expected values of the functions below. (Do not simplify fractions.)

- $E[X] = -1\cdot\frac{7}{20} + 0\cdot\frac{8}{20} + 1\cdot\frac{5}{20} = \boxed{\frac{-2}{20}}$
- $E[Y] = -1\cdot\frac{8}{20} + 0\cdot\frac{5}{20} + 1\cdot\frac{7}{20} = \boxed{\frac{-1}{20}}$
- $E[X^2] = (-1)^2\cdot\frac{7}{20} + 0^2\cdot\frac{8}{20} + 1^2\cdot\frac{5}{20} = \boxed{\frac{12}{20}}$
- $E[Y^2] = (-1)^2\cdot\frac{8}{20} + 0^2\cdot\frac{5}{20} + 1^2\cdot\frac{7}{20} = \boxed{\frac{15}{20}}$
- $E[XY] = (-1)(-1)\cdot\frac{3}{20} + (-1)(1)\cdot\frac{1}{20} + (1)(-1)\cdot\frac{2}{20} + (1)(1)\cdot\frac{2}{20} = \boxed{\frac{2}{20}}$

  Ignore terms where $xy=0$

- $E[X+Y] = (-2)\left(\frac{3}{20}\right) + (-1)\left(\frac{3}{20}+\frac{3}{20}\right) + (0)\left(\frac{1}{20}+\frac{1}{20}+\frac{2}{20}\right) + (1)\left(\frac{4}{20}+\frac{1}{20}\right) + (2)\left(\frac{2}{20}\right) = \boxed{\frac{-3}{20}}$

  Organize terms by $x+y$ – diagonals!

Compute the variance, covariance, and correlation of X and Y. **(Do not simplify fractions!)**

- $\mathrm{Var}[X] = E[X^2] - (E[X])^2 = \frac{12}{20} - \frac{4}{400} = \boxed{\frac{236}{400}}$
- $\sigma_X = \sqrt{\mathrm{Var}[X]} = \sqrt{\frac{236}{400}} = \boxed{\frac{\sqrt{236}}{20}}$
- $\mathrm{Var}[Y] = E[Y^2] - (E[Y])^2 = \frac{15}{20} - \frac{1}{400} = \boxed{\frac{299}{400}}$
- $\sigma_Y = \sqrt{\mathrm{Var}[Y]} = \sqrt{\frac{299}{400}} = \boxed{\frac{\sqrt{299}}{20}}$
- $\mathrm{Cov}[X,Y] = E[X\cdot Y] - E[X]\cdot E[Y] = \frac{2}{20} - \frac{2}{400} = \boxed{\frac{38}{400}}$
- $\rho_{XY} = \frac{\mathrm{Cov}[X,Y]}{\sigma_X\cdot\sigma_Y} = \boxed{\frac{38}{\sqrt{236}\cdot\sqrt{299}}}$